Problem Sheet 6 This is a set of questions on the material covered in lectures in the last week of the semester. For this reason there will be no supervision in which to go over the answers. Nonetheless, as an important part of your revision you should work through these questions and compare your answers to those available from my web site.

1. Count the number of elements in the following subsets of S_5.

   (i) The set of cycles of length 2.

   (ii) The set of cycles of length 3.

   (iii) The set of permutations that fix a given element?

   Which of the above subsets are closed under composition?

2. Calculate the orders of the following permutations in S_{11} :

   (i)
   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 5 & 10 & 11 & 7 & 4 & 9 & 1 & 2 & 3 & 6 \end{pmatrix},$$

   (ii)
   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 4 & 6 & 8 & 10 & 5 & 7 & 9 & 11 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 & 11 \end{pmatrix}.$$

3. Calculate the orders of the following permutations in S_5 :

   (i) $(1,2,3) \circ (1,3,4) \circ (1,3,5)$,

   (ii) $(1,2) \circ (1,3) \circ (1,4) \circ (1,5)$,

   (iii) $(2,3,5) \circ (1,2) \circ (2,4) \circ (1,2)$.

4. a) Write the five permutations in Question 7 of Sheet 5 as products of distinct cycles.

   b) Calculate the orders of these permutations.

5. Give an example of a permutation in S_{13} with the largest possible order.

6. Define $*$ on $\mathbb{Q}$ by
$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d},$$
where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$. Is this a binary operation?

7. Are the following operations closed?

   (i) Addition on the set of odd integers,

   (ii) Multiplication on the set of even integers,

   (iii) $a \circ b = a + b - ab$ on the set of odd integers.

8. Which of the following binary operations are associative and which are commutative? **Give your reasons**.

   (i) $x * y = 2(x + y)$ on $\mathbb{R}$,

   (ii) $x * y = x|y|$ on $\mathbb{R}$,

   (iii) $x * y = \dfrac{x+y}{xy}$ on $\mathbb{R} \setminus \{0\}$,

   (iv) $x * y = x + y - xy$ on $\mathbb{Z}$,

   (v) $x * y = \max(x, y)$ on $\mathbb{N}$,

9. Which of the following binary operations have identities

   (i) $x * y = \max(x, y)$ on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

   (ii) $x * y = \max(x, y)$ on $\mathbb{Z}$,

   (iii) $x * y = x + y - xy$ on $\mathbb{Q}$,

   (iv) matrix multiplication on the set
$$\left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}.$$

10. Why is $(\mathbb{Z}, -)$ not a group?

11. (a) Draw up the multiplication tables for the following sets with binary operations.

    (i) $(\{1, 4, 7, 13\}, \times_{15})$,

    (ii) $(\{3, 6, 9, 12\}, \times_{15})$,

    (iii) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

    with matrix multiplication.

    (b) In each example above find the identity element and list the inverses of each element.

12. a) Draw up the multiplication table for

$$(\{4, 8, 12, 16, 20, 24\}, \times_{28}).$$

    Find the identity element and the inverse of each element.

    b) Can you find a **proper** subset of $\{4, 8, 12, 16, 20, 24\}$ which is closed under $\times_{28}$?